Correction Model Resit Lin Alg 2
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1 a) $\tilde{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\tilde{x} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Hence $\langle 1, x \rangle = 1.0 + 1.1 + 1(-1) = 0$

b) $\mathcal{E} = \{ p_0 + p_2 x^2 \mid p_0, p_2 \in \mathbb{R} \}$

$\mathcal{O} = \{ q_1 x \mid q_1 \in \mathbb{R} \}$

Now let $p(x) = p_0 + p_2 x^2$. Then

$\tilde{p} = \begin{pmatrix} p_0 \\ p_0 + p_2 \\ p_0 + p_2 \end{pmatrix}$. For $q(x) = q_1 x$

we have $\tilde{q} = \begin{pmatrix} 0 \\ q_1 \\ -q_1 \end{pmatrix}$

Thus, $\langle p(x), q(x) \rangle = \tilde{p}^T \tilde{q} = 0$ as desired

c) An obvious basis for $\mathcal{E}$ is $\{1, x^2\}$

Since $\tilde{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\tilde{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, this basis is not even orthogonal. Thus we have to construct a different basis $\{e_1, e_2\}$

Since $\|1\|^2 = \tilde{1}^T \tilde{1} = 3$, we will take

$$e_1(x) = \frac{1}{\sqrt{3}}$$

Then $\tilde{e}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ so $\|e_1\|^2 = \tilde{e}_1^T \tilde{e}_1 = 1$

and hence $\|e_1\| = 1$.

For a second basis vector, try first

$$p(x) = a + bx^2.$$

Then $\tilde{p} = \begin{pmatrix} a \\ a+b \\ a+b \end{pmatrix}$. We want $\tilde{e}_1^T \tilde{p} = 0$

and this requires $a + (a+b) + (a+b) = 0$

so $3a + 2b = 0$. Take $a = 1$, $b = -3/2$

Then

$$p(x) = 1 - \frac{3}{2} x^2$$

is orthogonal to $e_1(x)$. For this $p(x)$ we have

$$\tilde{p} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

so $\|p\|^2 = \tilde{p}^T \tilde{p} = 1 + \frac{1}{4} + \frac{1}{4} = \frac{3}{2}$

and therefore $\|p\| = \sqrt{3}/\sqrt{2}$

Finally, take

$$e_2(x) = \frac{p(x)}{\|p\|} = \frac{\sqrt{2}}{\sqrt{3}}\left(1 - \frac{3}{2}x^2\right)$$

so

$$e_2(x) = \frac{\sqrt{2}}{\sqrt{3}} - \frac{\sqrt{3}}{\sqrt{2}}x^2$$

2) a) We have $Av = \lambda v$, $v \neq 0$

For any power A^k of A we have
$A^k v = \lambda^k v$.

Now suppose $p(s) = p_0 + p_1 s + \dots + p_k s^k$

Then

$$p(A)v = p_0 + p_1 A v + p_2 A^2 v + \dots + p_k A^k v$$

$$= (p_0 + p_1\lambda + p_2\lambda^2 + \dots + p_k\lambda^k)v$$

$$= p(\lambda)v.$$

Thus v is an eigenvector of $p(A)$ with corresponding eigenvalue $p(\lambda)$.

b) Now suppose $p(A) = 0$.

Let λ be an eigenvalue of A. Then $Av = \lambda v$ for some $v \neq 0$.
According to a) we have

$$0 = p(A)v = p(\lambda)v$$

Since $v \neq 0$ we must have $p(\lambda) = 0$.

3) a) We need to prove $x^T M^T A M x > 0$ for all $x \neq 0$. So: let $x \neq 0$. Since M is nonsingular, also $y := Mx \neq 0$.
Since $A > 0$ we have $y^T M y > 0$. Thus

$$x^T M^T A M x = y^T A y > 0.$$

b) Since A is square, it suffices to show that $Ax = 0$ implies $x = 0$.
So: let $Ax = 0$. Then $x^T A x = 0$.
Since $A > 0$ this implies $x = 0$ (since $x^T A x > 0$ for all $x \neq 0$)

c) We need to show that $x_2^T A_{22} x_2 > 0$ for all $x_2 \neq 0$ in $\mathbb{R}^{n-k}$.
Let $x_2 \neq 0$. Define $x \in \mathbb{R}^n$ by $x = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$
Then $x \neq 0$. Hence $x^T A x > 0$.
Thus

$$x_2^T A_{22} x_2 = x^T A x > 0.$$

d) This product is equal to

$$\begin{pmatrix} A_{11} - A_{12} A_{22}^{-1} A_{12}^{T} & O \\ O & A_{22} \end{pmatrix}$$

e) ($\Rightarrow$) $A > 0$ implies that the product (1) is also > 0 (this follows from a)) Then, by c), we have $A_{11} - A_{12} A_{22}^{-1} A_{12}^{T} > 0$ and $A_{22} > 0$

($\Leftarrow$) If $A_{22} > 0$ and $A_{11} - A_{12} A_{22}^{-1} A_{12}^{T} > 0$ then obviously the blockdiagonal matrix obtained as (1) is > 0. By (a) we then find $A > 0$.

4) a) We need to prove that $A^H A = A A^H$.

It is given that $U^H A U = \Lambda$ diagonal.
Thus $A = U \Lambda U^H$. Therefore,

$$\begin{aligned} A^H A &= (U \Lambda U^H)^H U \Lambda U^H \\ &= U \Lambda^H U^H U \Lambda U^H \\ &= U \Lambda^H \Lambda U^H. \end{aligned}$$

$$\begin{aligned} A A^H &= U \Lambda U^H U \Lambda^H U^H \\ &= U \Lambda \Lambda^H U^H. \end{aligned}$$

These are indeed equal since Λ is diagonal: if

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

then $\Lambda^H = \begin{pmatrix} \bar{\lambda}_1 & & \\ & \ddots & \\ & & \bar{\lambda}_n \end{pmatrix}$

so $\Lambda^H \Lambda = \begin{pmatrix} |\lambda_1|^2 & & \\ & \ddots & \\ & & |\lambda_n|^2 \end{pmatrix} = \Lambda \Lambda^H$.

b) T is square upper triangular

Let $T = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ & t_{22} & \ddots & \vdots \\ & 0 & \ddots & t_{n-1,n} \\ & & & t_{nn} \end{pmatrix}$

($\Rightarrow$) Assume T normal, i.e. $T^H T = T T^H$.

$T^H = \begin{pmatrix} \bar{t}_{11} & & & 0 \\ \bar{t}_{12} & \bar{t}_{22} & & \\ \vdots & & \ddots & \\ \bar{t}_{1n} & & & \bar{t}_{nn} \end{pmatrix}$

Now, $(T T^H)_{11} = |t_{11}|^2 + |t_{12}|^2 + \dots + |t_{1n}|^2$

and $(T^H T)_{11} = |t_{11}|^2$.

This implies $t_{12} = 0, t_{13} = 0, \dots, t_{1n} = 0$

Next, $(T T^H)_{22} = |t_{22}|^2 + |t_{23}|^2 + \dots + |t_{2n}|^2$